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On the Saturation Order of Approximation Processes Involving Jacobi Polynomials

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INTRODUCTION

We are concerned with the saturation order of an approximation kernel originated by a matrix transform of Fourier–Jacobi expansion. Matrix transforms turn out to be summation methods under various conditions. We start with a triangular matrix $\Lambda = ((\lambda_{n,k}))$, $n = 0, 1, 2, ..., k = 0, 1, 2, ...,$ such that $\lambda_{n,k} = 0$ for $k > n$ and prove a general result on saturation problems involving approximation by Jacobi polynomials.

1

Preliminaries. Let X denote one of the function spaces C or $L^p(1 \leqslant p \leqslant \infty)$, on $[-1, 1]$. The space C is a Banach space with norm

$$\| f \|_X = \sup_{0 \leqslant \theta \leqslant \pi} | f(\cos \theta)|, \qquad f \in X = C, \tag{1.1}$$

and $X = L^p(1 \leqslant p < \infty)$, with weight function

$$\rho^{(\alpha,\beta)}(\theta) = (\sin \theta/2)^{2\alpha+1} (\cos \theta/2)^{2\beta+1}, \qquad \alpha, \beta > -1, \tag{1.2}$$

is a Banach space with norm

$$\| f \|_X \equiv \| f \|_p = \left\{ \int_0^\pi | f(\cos \theta)|^p \rho^{(\alpha,\beta)}(\theta)\, d\theta \right\}^{1/p}_{f \in X = L^p}. \tag{1.3}$$

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Also $X = L^\infty$ is a Banach space if endowed with the norm

$$\| f \|_X = \| f \|_\infty = \operatorname{ess} \sup_{0 \leqslant \theta \leqslant \pi} | f(\cos \theta)|_{f \in X = L^\infty}. \tag{1.4}$$

With $f \in X$, we associate the Fourier–Jacobi expansion

$$f(\cos \theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos \theta), \tag{1.5}$$

where

$$\begin{aligned} \omega_n^{(\alpha,\beta)} &= \left(\int_0^\pi \{ R_n^{(\alpha,\beta)}(\cos \theta) \}^2 \rho^{(\alpha,\beta)}(\theta)\, d\theta \right)^{-1} \\ &= \frac{(2n+\alpha+\beta+1)\, \Gamma(n+\alpha+\beta+1)\, \Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\, \Gamma(n+1)\, \Gamma(\alpha+1)\, \Gamma(\alpha+1)} \\ &= n^{2\alpha+1} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}. \end{aligned} \tag{1.6}$$

$f^\wedge(n)$ is the Fourier-Jacobi transform of f given by

$$f^\wedge(n) = \int_0^\pi f(\cos \theta)\, R_n^{(\alpha,\beta)}(\cos \theta)\, \rho^{(\alpha,\beta)}(\theta)\, d\theta, \tag{1.7}$$

where

$$R_n^{(\alpha,\beta)}(\cos \theta) = P_n^{(\alpha,\beta)}(\cos \theta) / P_n^{(\alpha,\beta)}(1).$$

$P_n^{(\alpha,\beta)}(x)$ is the nth Jacobi polynomial of order (α, β) and degree n. The generalized translate of f with expansion (1.5), introduced by Askey and Wainger [1], is $T_\phi f$ with expansion

$$T_\phi f(\cos \theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos \theta)\, R_n^{(\alpha,\beta)}(\cos \phi). \tag{1.8}$$

It is known that T_ϕ is a positive operator for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ and has operator norm 1 (see Gasper [6]). For $f_1, f_2 \in X$, the convolution $f_1 * f_2$ is defined (see Askey and Wainger [1]) by

$$f_1 * f_2(\cos \theta) = \int_0^\pi T_\phi f_1(\cos \theta) f_2(\cos \phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi, \tag{1.9}$$

and has the following properties. Let $f_1, f_2, f_3 \in L^1$ and $g \in X$. Then

$$\begin{aligned} &\text{(i)} && f_1 * f_2 = f_2 * f_1 \\ &\text{(ii)} && f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3 \\ &\text{(iii)} && \| f_1 * g \|_X \leqslant \| f_1 \|_1 \, \| g \|_X \\ &\text{(iv)} && (f_1 * f_2)^\wedge (n) = f_1^\wedge (n) f_2^\wedge (n). \end{aligned} \tag{1.10}$$

Furthermore, a positive summability kernel, a quasi-positive kernel, and an approximation kernel have been defined by Bavinck along the lines of Butzer and Berens [5] which is a landmark in the sphere of abstract approximation theory and its various applications. One of the important achievements of Bavinck is that the convolution of any approximation kernel (satisfying his definition) with an element of X yields a strong approximation process in X. Our approximation kernel is different from that but it leads to the process of strong approximation in X. In the next section we denote by X either C or L^p $(1 \leqslant p < \infty)$ with $\alpha \geqslant \beta \geqslant -\frac{1}{2}$.

2

Applications. The modulus of continuity $\omega(\phi, f; X)$ in X is defined by Bavinck [2] as

$$\omega(\phi, f; X) = \omega(\phi) \stackrel{\text{def}}{=} \sup_{0 \leqslant \psi \leqslant \phi} \| T_\psi f(\cdot) - f(\cdot) \|_{\phi \geqslant 0, f \in X}. \tag{2.1}$$

Also $f \in \operatorname{Lip}(\gamma, X)$, $0 < \gamma \leqslant 2$, if there exists a positive c such that

$$\omega(\phi, f; X) \leqslant c\phi^\gamma. \tag{2.2}$$

The subspace $\operatorname{Lip}(\gamma, X)$ of X is a Banach space if endowed with the norm

$$\| f \|_{\operatorname{Lip}(\gamma, X)} = \| f \|_X + \sup_{n \in Z^+} (n^\gamma \omega(n^{-1}, f; X)) \tag{2.3}$$

and properties of $\omega(\phi, f; X)$ anologous to those of the classical modulus of continuity that have been given by Bavinck [2].

We have (see Askey and Wainger [1] and Gasper [6])

$$\begin{aligned} &R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi) \\ &\quad = \int_0^\pi R_n^{(\alpha,\beta)}(\cos\psi)\, K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi, \end{aligned} \tag{2.4}$$

where $K(\cos\theta, \cos\phi, \cos\psi) \geqslant 0$ is a symmetric function, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, and

$$\int_0^\pi K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi = 1. \tag{2.5}$$

Thus the generalized translate of $f(\cos\theta) \in L^1$ which has the Fourier–Jacobi expansion (1.5), can be defined as

$$\begin{aligned} T_\phi f(\cos\theta) &\overset{\text{def}}{=} f(\cos\theta, \cos\phi) \\ &= \int_0^\pi f(\cos\psi)\, K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi \\ &\sim \sum_{n=0}^{\infty} f^\wedge(n)\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi). \end{aligned} \tag{2.6}$$

We denote the nth partial sum of (1.5) by $s_n(f, \cos\theta)$ so that, by orthogonality of $R_n^{(\alpha,\beta)}(\cos\theta)$, we get (cf. Szegö [11, (4.5.3)]

$$\begin{aligned} s_n(f, \cos\theta) - f(\cos\theta) &= \sum_{\nu=0}^{n} \int_0^\pi [f(\cos\phi) - f(\cos\theta)]\, \omega_\nu^{(\alpha,\beta)} \\ &\quad \times R_\nu^{(\alpha,\beta)}(\cos\theta)\, R_\nu^{(\alpha,\beta)}(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi \\ &= \sum_{\nu=0}^{n} \omega_\nu^{(\alpha,\beta)} \int_0^\pi [f(\cos\phi) - f(\cos\theta)]\, \rho^{(\alpha,\beta)}(\phi) \\ &\quad \times \left\{ \int_0^\pi K(\cos\theta, \cos\phi, \cos\psi)\, R_\nu^{(\alpha,\beta)}(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi) \right\} d\phi \\ &= \int_0^\pi [f(\cos\phi) - f(\cos\theta)]\, \rho^{(\alpha,\beta)}(\phi) \int_0^\pi K(\cos\theta, \cos\phi, \cos\psi) \\ &\quad \times \sum_{\nu=0}^{n} \omega_\nu^{(\alpha,\beta)} R_\nu^{(\alpha,\beta)}(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi\, d\phi \\ &= L_n \int_0^\pi R_n^{(\alpha+1,\beta)}(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi) \int_0^\pi [f(\cos\phi) - f(\cos\theta)] \\ &\quad \times K(\cos\theta, \cos\phi, \cos\psi)\, \rho^{(\alpha,\beta)}(\phi)\, d\phi\, d\psi \end{aligned}$$

using the symmetric property of $K(\cos\theta, \cos\phi, \cos\psi)$. Here

$$L_n = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(1) = \frac{(\alpha+1)\,\omega_n^{(\alpha+1,\beta)}}{2n+\alpha+\beta+2}$$

$$= n^{2\alpha+2}\{1+O(1/n)\} \stackrel{\text{def}}{=} n^{2\alpha+2}L(n). \tag{2.7}$$

$L(n)$ is a slowly varying function of n in the sense of Bavinck [3] such that $L(n) \to 1$ and, given $\varepsilon, \delta > 0$, there exist n_1, n_2 so that $n^{\varepsilon}L(n)$ is increasing for $n > n_1$ and $n^{-\delta}L(n)$ is decreasing for $n > n_2$. Now

$$s_n(f, \cos\theta) - f(\cos\theta)$$

$$= L_n \int_0^{\pi} [T_\psi f(\cos\theta) - f(\cos\theta)]\, R_n^{(\alpha+1,\beta)}(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi. \tag{2.8}$$

Let $\Lambda = ((\lambda_{n,k}))$ be a triangular matrix with $\lambda_{n,0} = 1$ for all n. Then the Λ-transform of series (1.5) is defined as

$$\sigma_n^{(\Lambda)}(f, \cos\theta) = \sum_{k=0}^{n} \Delta\lambda_{n,k} s_k(f, \cos\theta), \qquad \Delta\lambda_{n,k} = \lambda_{n,k} - \lambda_{n,k+1}. \tag{2.9}$$

By (2.8),

$$\sigma_n^{(\Lambda)}(f, \cos\theta) - f(\cos\theta)$$

$$= \int_0^{\pi} [T_\psi f(\cos\theta) - f(\cos\theta)]\, K_n^{(\Lambda)}(\cos\psi)\, \rho^{(\alpha,\beta)}(\psi)\, d\psi, \tag{2.10}$$

where

$$K_n^{(\Lambda)}(\cos\psi) \stackrel{\text{def}}{=} K_n^{(\Lambda)}(\psi) = \sum_{k=0}^{n} \Delta\lambda_{n,k} L_k R_k^{(\alpha+1,\beta)}(\cos\psi). \tag{2.11}$$

3

The transform $\sigma_n^{(\Lambda)}(f, \cos\theta)$ is also called the Λ-mean of (1.5). The necessary and sufficient condition for the regularity of the Λ-method of summation is

$$\lim_{n\to\infty} \Delta\lambda_{n,k} = 0 \qquad \text{for} \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

For $\Delta\lambda_{n,k} = p_{n-k}/P_n$, $P_n = p_0 + p_1 + \cdots + p_n$, $p_0 > 0$, the method Λ reduces to the (N, P_n) method which is a generalization of Cesàro's method. On the other hand, if we take $\Delta\lambda_{n,k} = p_k/P_n$ ($\{p_k\}$ non-negative, non-decreasing), the Λ-method reduces to the $(\bar{N}, p_n)$ method. We consider a non-negative, non-decreasing sequence $\{\Delta\lambda_{n,k}\}$ arising out of a lower triangular matrix $((\lambda_{n,k}))$ (with $\lambda_{n,0}=1$ for all n), such that, for $k=0, 1, 2, \ldots,$ and $i=0, 1, \ldots,$

$$\frac{\Delta\lambda_{n,k}}{\Delta\lambda_{n,i}} \leqslant M \qquad \text{for} \quad k \geqslant i, \tag{3.2}$$

where M is any fixed positive number.

Now we intend to prove the following theorem where A is an absolute constant, not the same at each occurrence.

THEOREM 1. *Let $\{\Delta\lambda_{n,k}\}$ be a non-negative, non-decreasing sequence with respect to k, satisfying* (3.2). *Let $\omega(\phi)$ be the modulus of continuity of $f \in X$. Then the saturation order of the kernel $K_n^{(\Lambda)}$ of the Λ-method of summation is given by*

$$\begin{aligned} &\|\sigma_n^{(\Lambda)}(f, \cos\theta) - f(\cos\theta)\|_X \\ &\quad = \|(f * K_n^{(\Lambda)})(\cdot) - f(\cdot)\|_X \\ &\quad \leqslant A\left(\sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v)\right) \\ &\qquad + A_n, \end{aligned} \tag{3.3}$$

where

$$A_n = An^{-2\beta-2}. \tag{3.4}$$

The saturation class or Favard's class $F(X, \sigma^{(\Lambda)})$ is the space of all $f \in X$ for which the right side of (3.3) *tends to zero as n tends to infinity and $F(X, \sigma^{(\Lambda)}) \subset X$.*

Further, if there exist positive numbers M_1, M_2 such that, for a large enough positive integer n,

$$\omega(1/(k+1)) \geqslant \begin{cases} M_1(k+1)^{\alpha-2\beta-3/2}\,(n-k+1)^{-\alpha-1/2} \\ \text{or} \\ M_2(k+1)^{\alpha-2\beta-1/2}\,(n-k+1)^{-\alpha-3/2}, \end{cases} \tag{3.5}$$

for $k=0, 1, 2, \ldots, k \leqslant n$, and $\alpha \geqslant \beta \geqslant -1/2$, then we have the following important result.

THEOREM 2. *Let $\{\Delta\lambda_{n,k}\}$ and $\omega(\phi)$ be given as in Theorem* 1 *and let* (3.2) *and* (3.5) *be satisfied. Then we have*

$$\begin{aligned}
&\| \sigma_n^{(A)}(f, \cos\theta) - f(\cos\theta) \|_X \\
&\quad = \| (f * K_n^{(A)})(\cdot) - f(\cdot) \|_X \\
&\quad \leqslant A \left(\sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \sum_{v=0}^{k} \Delta\lambda_{n,n-v} (n-v+1)^{\alpha+1/2} L(n-v) \right) \qquad (3.6)
\end{aligned}$$

and the saturation class or Favard's class $F(X, \sigma^{(A)}) \subset X$ *is the space of all* $f \in X$ *for which the right side of* (3.6) *tends to zero as n tends to infinity.*

Remark. As pointed out in [3], many classical results [7, 8, 10] for Fourier series are carried over by (3.3) and (3.6) in a powerful way.

In the proof we use following results.

LEMMA 1 [9]. *If* $\{\Delta\lambda_{n,k}\}$ *is non-negative, non-decreasing with respect to* k, *then, for* $0 \leqslant a \leqslant b \leqslant \infty$, $0 < t \leqslant \pi$, *and any* n,

$$\left| \sum_{k=a}^{b} \Delta\lambda_{n,n-k} e^{i(n-k)t} \right| \leqslant (2\pi+1) \sum_{k=0}^{\tau} \Delta\lambda_{n,n-k}, \qquad (3.7)$$

where $\tau = [1/t]$.

LEMMA 2. *We have*

$$A_n \leqslant A \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \sum_{v=0}^{k} \Delta\lambda_{n,n-v} (n-v+1)^{\alpha+1/2} L(n-v) \qquad (3.8)$$

provided (3.5) *is satisfied.*

Proof. By (3.4), we have

$$\begin{aligned}
A_n &= A n^{-2\beta-2} \sum_{v=0}^{n} \Delta\lambda_{n,n-v} \qquad (\text{for } \lambda_{n,0} = 1 \quad \text{and} \quad \lambda_{n,n+1} = 0) \\
&\leqslant A \sum_{v=0}^{n} \Delta\lambda_{n,n-v} \sum_{k=v+1}^{n} k^{-2\beta-3} \\
&\leqslant A \sum_{v=0}^{n} \Delta\lambda_{n,n-v} (n-v+1)^{\alpha+1/2} L(n-v) \\
&\quad \times \sum_{k=v+1}^{n} k^{-2\beta-3} (n-k+1)^{-\alpha-1/2} \\
&\quad \left(\text{for} \quad \alpha \geqslant -\frac{1}{2} \quad \text{and} \quad L(n-v) = O(1) \right)
\end{aligned}$$

$$= A \sum_{v=0}^{n} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v)$$

$$\times \sum_{k=v+1}^{n} \frac{\omega(1/k)}{k^{\alpha+3/2}} \left[\frac{k^{\alpha-2\beta-3/2}(n-k+1)^{-\alpha-1/2}}{\omega(1/k)} \right]$$

$$= A \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}}$$

$$\times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v) \qquad \text{(by the first part of (3.5))}.$$

Here we used the fact that

$$\sum_{k=v+1}^{n} k^{-2\beta-3} = A(v+1)^{-2\beta-2}.$$

As

$$(n-v+1)^{-1} \sum_{k=v+1}^{n} k^{-2\beta-2} \geqslant (n-v+1)^{-1} (n-v+1)\, n^{-2\beta-2} = n^{-2\beta-2},$$

we have

$$A_n \leqslant A \sum_{v=0}^{n} \Delta\lambda_{n,n-v}(n-v+1)^{-1}$$

$$\times \sum_{k=v+1}^{n} k^{-2\beta-2}(n-k+1)^{\alpha+3/2} (n-k+1)^{-\alpha-3/2}$$

$$= A \sum_{v=0}^{n} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v)$$

$$\times \sum_{k=v+1}^{n} \frac{\omega(1/k)}{k^{\alpha+3/2}} \left[\frac{k^{\alpha-2\beta-1/2}(n-k+1)^{-\alpha-3/2}}{\omega(1/k)} \right]$$

$$= A \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{\alpha+3/2}}$$

$$\times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v),$$

by the second part of (3.5). This proves the lemma.

LEMMA 3. *If* $\{\Delta\lambda_{n,k}\}$ *is non-negative, non-decreasing, and* $\omega(\phi)$ *is the modulus of continuity defined by* (2.1), *then, for all positive integers n and for* $\alpha \geqslant -\frac{1}{2}$, *we have*

$$n^{-2\alpha-2}\omega(1/n)\sum_{v=0}^{n}\Delta\lambda_{n,n-v}L_{n-v}$$
$$\leqslant A\sum_{k=0}^{n}\frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}}$$
$$\times\sum_{v=0}^{k}\Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}L(n-v). \tag{3.9}$$

Proof.

$$\sum_{k=0}^{n}\frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}}\sum_{v=0}^{k}\Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}L(n-v)$$
$$=\sum_{v=0}^{n}\Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}L(n-v)\sum_{k=v}^{n}\frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}}$$
$$\geqslant\sum_{v=0}^{n}\Delta\lambda_{n,n-v}L(n-v)(n-v+1)^{\alpha+3/2}\frac{\omega(1/n)}{n^{3/2+\alpha}}$$
$$\geqslant n^{-\alpha-3/2}\omega(1/n)\sum_{v=0}^{n}\Delta\lambda_{n,n-v}L(n-v)(n-v+1)^{\alpha+3/2}$$
$$=n^{\alpha+1/2}n^{-2\alpha-2}\omega(1/n)\sum_{v=0}^{n}\Delta\lambda_{n,n-v}L_{n-v}(n-v+1)^{-\alpha-1/2}$$
$$\geqslant n^{-2\alpha-2}\omega(1/n)\sum_{v=0}^{n}\Delta\lambda_{n,n-v}L_{n-v}$$
$$(\text{for } L_{n-v}=(n-v+1)^{2\alpha+2}L(n-v)).$$

QED

Besides these lemmas, we use results from [11] without special reference. Thus, with the help of Szegö's Hilb-type asymptotic formula, we make use of the following estimate (see Bingham [4]) for $c/n\leqslant\theta\leqslant\pi-c/n$, where n is large enough and $\alpha\geqslant\beta\geqslant-\frac{1}{2}$:

$$\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)$$
$$=\frac{2^{3/2}}{\pi^{1/2}\Gamma(\alpha+1)}\frac{n^{\alpha+1/2}\cos\{n\theta+(\alpha+\beta+1)\,\theta/2\}}{(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta}}[1+O(1/n)]$$
$$=\frac{2^{3/2}}{\pi^{1/2}\Gamma(\alpha+1)}n^{\alpha+1/2}\sin^{-\alpha-1/2}\theta/2\cos^{-\beta}\theta/2$$
$$\times\cos\{n\theta+(\alpha+\beta+1)\,\theta/2\}\,L(n). \tag{3.10}$$

Proof of Theorem 1. We have, from (2.10),

$$\sigma_n^{(A)}(f,\cos\theta)-f(\cos\theta)=(f*K_n^{(A)})(\cos\theta)-f(\cos\theta)$$

as our result shows that change of order of summation is justified. Thus

$$\begin{aligned}
&\| \sigma_n^{(\Lambda)}(f, \cos\theta) - f(\cos\theta)\|_X \\
&\leqslant \int_0^\pi \| T_\psi f(\cdot) - f(\cdot)\|_X \, | K_n^{(\Lambda)}(\psi)| \, \rho^{(\alpha,\beta)}(\psi)\, d\psi \\
&= \int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi-\pi/(n+1)} + \int_{\pi-\pi/(n+1)}^{\pi} \\
&= P + Q + R \text{ (say)}.
\end{aligned}$$

But

$$\begin{aligned}
P &\leqslant \int_0^{\pi/(n+1)} \omega(\psi, f; X) | K_n^{(\Lambda)}(\psi)| \, \rho^{(\alpha,\beta)}(\psi)\, d\psi \\
&= A\omega(\pi/(n+1))(n+1)^{-2\alpha-2} \sum_{k=0}^{n} \Delta\lambda_{n,k} L_k \\
&= A\omega(1/n)\, n^{-2\alpha-2} \sum_{v=0}^{n} \Delta\lambda_{n,n-v} L_{n-v} \\
&\leqslant A \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \\
&\quad \times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v) \qquad \text{(by Lemma 3).} \qquad (3.11)
\end{aligned}$$

Also, let there exist an n_0 such that, for $n > n_0$, the estimate (3.10) holds. Then

$$\begin{aligned}
Q &\leqslant \int_{\pi/(n+1)}^{\pi-\pi/(n+1)} \omega(\psi, f; X) \left| \sum_{k=0}^{n_0} \Delta\lambda_{n,k} L_k R_k^{(\alpha+1;\beta)}(\cos\psi) \right| \rho^{(\alpha,\beta)}(\psi)\, d\psi \\
&\quad + \int_{\pi/(n+1)}^{\pi-\pi/(n+1)} \omega(\psi, f; X) \left| \sum_{k=n_0+1}^{n} \Delta\lambda_{n,k} L_k R_k^{(\alpha+1,\beta)}(\cos\psi) \right| \rho^{(\alpha,\beta)}(\psi)\, d\psi \\
&= Q_1 + Q_2 \text{ (say)},
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &= A \int_{\pi/(n+1)}^{\pi} \psi^{2\alpha+1} \omega(\psi, f; X) \sum_{k=0}^{n_0} \Delta\lambda_{n,k}\, d\psi \\
&= An_0 \int_{\pi/(n+1)}^{\pi} \psi^{2\alpha+1} \omega(\psi, f; X)(\Delta\lambda_{n,n_0})\, d\psi
\end{aligned}$$

$$\leqslant A n_0 \Delta\lambda_{n,n_0} \sum_{k=0}^{n} \int_{\pi/(k+2)}^{\pi/(k+1)} \psi^{2\alpha+1}\omega(\psi)\, d\psi$$

$$\times \frac{(k+1)\,\Delta\lambda_{n,n-k}(n-k+1)^{\alpha+1/2}\, L(n-k)}{(k+1)\,\Delta\lambda_{n,n-k}(n-k+1)^{\alpha+1/2}\, L(n-k)}$$

$$= A n_0 \Delta\lambda_{n,n_0} \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \frac{(k+1)^{-5/2-\alpha}}{\Delta\lambda_{n,n-k}(n-k+1)^{\alpha+1/2}}$$

$$\times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}\, L(n-v)$$

$$= A\left[\frac{\Delta\lambda_{n,n_0}}{\Delta\lambda_{n,0}}\right] \sum_{k=0}^{n} \frac{\omega(1/k+1)}{(k+1)^{3/2+\alpha}}$$

$$\times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}\, L(n-v)$$

(for $\{\Delta\lambda_{n,k}\}\nearrow \Rightarrow \{\Delta\lambda_{n,n-k}\}\searrow \Rightarrow \{1/\Delta\lambda_{n,n-k}\}\nearrow$)

$$= A\left(\sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}}\right.$$

$$\left.\times \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2}\, L(n-v)\right) \qquad \text{(by (3.2))}.$$

Keeping in mind that $L(k)$ absorbs the error terms, we get, by (3.10), for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$,

$$Q_2 = \frac{2^{3/2}}{\pi^{1/2}\Gamma(\alpha+1)} \int_{\pi/(n+1)}^{\pi-\pi/(n+1)} \omega(\psi)(\sin\psi/2)^{\alpha-1/2}(\cos\psi/2)^{\beta+1}\, d\psi$$

$$\times \left| \sum_{k=n_0+1}^{n} \Delta\lambda_{n,k} \frac{k^{\alpha+3/2}L(k)}{2k+\alpha+\beta+2} \cos\{k\psi + (\alpha+\beta+2)\,\psi/2\} \right|$$

$$= \frac{2^{3/2}}{\pi^{1/2}\Gamma(\alpha+1)} \int_{\pi/(n+1)}^{\pi-\pi/(n+1)} \omega(\psi)\sin^{\alpha-1/2}\psi/2\,\cos^{\beta+1}\psi/2$$

$$\times \left| \sum_{v=0}^{n-n_0-1} \Delta\lambda_{n,n-v} \frac{(n-v)^{\alpha+3/2}\, L(n-v)}{2(n-v)+\alpha+\beta+2} \right.$$

$$\left. \times \cos\{(n-v)\,\psi + (\alpha+\beta+2)\,\psi/2\} \right| d\psi$$

$$\leqslant A \int_{\pi/(n+1)}^{\pi} \frac{\omega(\psi)}{\psi^{1/2-\alpha}}\, d\psi$$

$$\times \sum_{v=0}^{[1/\psi]} \Delta\lambda_{n,n-v} \frac{(n-v)^{\alpha+3/2}\, L(n-v)}{2(n-v)+\alpha+\beta+2}$$

(for the integrand is positive),

$$= A \sum_{k=0}^{n} \int_{\pi/(k+2)}^{\pi/(k+1)} \frac{\omega(\psi)}{\psi^{1/2-\alpha}} \sum_{v=0}^{[1/\psi]} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v)$$

$$\leqslant A \sum_{k=0}^{n} \frac{\omega(1/(k+1))}{(k+1)^{3/2+\alpha}} \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v).$$

Again,

$$R \leqslant \int_{\pi-\pi/(n+1)}^{\pi} \omega(\psi) | K_n^{(\Lambda)}(\psi)| \, \rho^{(\alpha,\beta)}(\psi)\, d\psi.$$

But

$$\begin{aligned} K_n^{(\Lambda)}(\psi) \equiv F(\cos\psi) &= \sum_{k=0}^{n} \Delta\lambda_{n,k} L_k R_k^{(\alpha+1,\beta)}(\cos\psi) \\ &= \Delta\lambda_{n,0} L_0 R_0^{(\alpha+1,\beta)}(\cos\psi) \\ &\quad + \sum_{k=1}^{n} b(k)\, k^{-1} \omega_k^{(\alpha+1,\beta)} R_k^{(\alpha+1,\beta)}(\cos\psi), \end{aligned}$$

where

$$b(k) = \frac{(\alpha+1)\,\Delta\lambda_{n,k}}{2+(\alpha+\beta+2)/k}, \qquad k = 1, 2, ..., n \geqslant k,$$

$b(k)$ being bounded and monotonic. Thus, applying the arguments of [3, 785–786] we see that F $(\cos\psi)$ is continuous in $0 < \psi \leqslant \pi$ and converges uniformly in n when ψ is very near to π, i.e., when n is large enough. In fact

$$\begin{aligned} K_n^{(\Lambda)}(\psi) &= \frac{(\alpha+1)\,\Delta\lambda_{n,0}}{\alpha+\beta+2}\, \omega_0^{(\alpha+1,\beta)} R_0^{(\alpha+1,\beta)}(\cos\psi) \\ &\quad + \frac{(\alpha+1)\,\Delta\lambda_{n,n}}{2+(\alpha+\beta+2)/n} (\log 11)^{-1} \\ &\quad \times \sum_{k=n_1}^{n_2} [\log(k+10)]\, k^{-1} \omega_k^{(\alpha+1,\beta)} R_k^{(\alpha+1,\beta)}(\cos\psi) \\ &\quad (1 \leqslant n_1 \leqslant n_2 \leqslant n), \end{aligned}$$

so that we can obviously apply the cited argument and our conclusion follows as $\Delta\lambda_{n,n} \leqslant M\Delta\lambda_{n,0} = M(1-\lambda_{n,1}) \geqslant 0$ since $\{\Delta\lambda_{n,k}\}$ are non-negative, non-decreasing, and satisfy (3.2). Hence

$$\begin{aligned} R &= A \int_0^{\pi/(n+1)} \psi^{2\beta+1}\, d\psi \\ &= A n^{-2\beta-2} = A_n. \end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

$$A_n \leqslant A\left(\sum_{k=0}^{n} \frac{\omega(1/k+1)}{(k+1)^{3/2+\alpha}} \sum_{v=0}^{k} \Delta\lambda_{n,n-v}(n-v+1)^{\alpha+1/2} L(n-v)\right)$$

by Lemma 2, under the present conditions. Thus this theorem follows by the proof of Theorem 1.

4

Corollary of Theorem 1.

COROLLARY 1. *Let* $\Delta\lambda_{n,k}=1/(n+1)$ *for* $k=0, 1, 2, \ldots$. *Then, if we denote the* $(C, 1)$ *mean of* (1.5) *by* $S_n^1(f, \cos\theta)$, *we have*

$$\begin{aligned} \| S_n^1(f, (\cdot)) - f(\cdot) \|_X &= \| (f * K_n^{(\Lambda)})(\cdot) - f(\cdot) \|_X \\ &\leqslant A\left[n^{\alpha-1/2} \sum_{k=0}^{n} \frac{\omega(1/k+1)}{(k+1)^{\alpha+1/2}} + n^{-2\beta-2} \right]. \end{aligned} \tag{4.1}$$

Proof. A particular case of the proof of Theorem 1.

Also, let us merely assume that $\Lambda=((\lambda_{n,k}))$ is a triangular matrix. Then

$$\begin{aligned} &\sigma_n^{(\Lambda)}(f, \cos\theta) - f(\cos\theta) \\ &\quad = \sum_{k=0}^{n} (k+1)\{S_k^1(f, \cos\theta) - f(\cos\theta)\}\, \Delta^2\lambda_{n,k}. \end{aligned} \tag{4.2}$$

Thus, by (4.1) and (4.2), we have the following general result.

THEOREM 3. *For any triangular matrix transformation of* (1.5), *we have*

$$\begin{aligned} &\| \sigma_n^{(\Lambda)}(f, (\cdot)) - f(\cdot) \|_X \\ &\quad = \| (f * K_n^{(\Lambda)})(\cdot) - f(\cdot) \|_X \\ &\quad \leqslant A \sum_{k=0}^{n} |\Delta^2\lambda_{n,k}| (k+1)^{\alpha+1/2} \\ &\qquad \times \left(\sum_{v=0}^{k} \frac{\omega(1/(v+1))}{(v+1)^{\alpha+1/2}} + (k+1)^{-\alpha-2\beta-3/2} \right), \end{aligned} \tag{4.3}$$

$$\Delta^2\lambda_{n,k} = \Delta\lambda_{n,k} - \Delta\lambda_{n,k+1},$$

for $\alpha \geqslant \beta \geqslant -\frac{1}{2}$. *The saturation class or Favard's class* $F(X, \sigma^{(\Lambda)}) \subset X$ *is the collection of all* $f \in X$ *such that the right side of* (4.3) *tends to zero as* $n \to \infty$.

The modulus sign on the right side of (4.3) may be removed by restricting $\{\Delta\lambda_{n,k}\}$. Inequality (4.3) seems to be the best possible but requires further investigation.

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